

Discrete Uniqueness Sets for Functions with Spectral Gaps

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Abstract

It is well-known that entire functions whose spectrum belongs to a fixed bounded set S , admit real uniformly discrete uniqueness sets Λ . We show that the same is true for much wider spaces of continuous functions. In particular, Sobolev spaces have this property whenever S is a set of infinite measure having "periodic gaps". The periodicity condition is crucial. For sets S with randomly distributed gaps, we show that the uniformly discrete sets Λ satisfy a strong non-uniqueness property: Every discrete function $c(\lambda) \in l^2(\Lambda)$ can be interpolated by an analytic L^2 -function with spectrum in S .

1 Introduction

Paley-Wiener Space. We will use the standard form of the Fourier transform:

$$F(t) = \hat{f}(t) := \int_{\mathbb{R}} e^{-2\pi itx} f(x) dx.$$

Given a measurable set S , the Paley-Wiener space PW_S consists of the inverse Fourier transforms of all square-integrable functions F which vanish a.e. outside S . The set S is called the spectrum of the space PW_S . Clearly, if the measure of S is finite, then $F \in L^1(\mathbb{R})$, and so every function $f \in PW_S$ is continuous. If S is bounded, then every $f \in PW_S$ is an entire function of exponential type.

Uniformly discrete sets. A set $\Lambda \subset \mathbb{R}$ is called *uniformly discrete* (u.d.) if $\delta(\Lambda) > 0$, where

$$\delta(\Lambda) := \inf_{\lambda, \lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| \quad (1)$$

is the infimal distance between different elements of Λ .

A u.d. set Λ is said to have the *uniform density* $D(\Lambda)$, if Λ is regularly distributed in the following sense:

$$\text{Card}(\Lambda \cap (x, x+r)) = rD(\Lambda) + o(r) \text{ uniformly on } x \text{ as } r \rightarrow \infty.$$

Uniqueness Problem. Let F be a space of continuous functions on the real line \mathbb{R} . A set $\Lambda \subset \mathbb{R}$ is called a *uniqueness set* for F if

$$f \in F, f|_{\Lambda} = 0 \Rightarrow f = 0.$$

Otherwise, Λ is called a *non-uniqueness set*.

Problem. *Which spaces of continuous functions on \mathbb{R} admit u.d. uniqueness sets?*

We will consider this problem for spaces of function whose spectrum belongs to a fixed set S . It is natural to distinguish between the following cases: S is a bounded set, an unbounded set of finite measure, and a set of infinite measure.

In the present paper we focus on spaces of continuous functions whose spectrum lies in a set S of infinite measure. In Sec. 3–5 we establish that wide spaces of such functions admit u.d. uniqueness sets, provided S has periodic gaps. The periodicity condition is important. In particular, in Sec. 6, for sets S with randomly distributed gaps we show that every u.d. set Λ satisfies some strong non-uniqueness property.

We start with a short survey of known results on the first two cases. A detailed discussion of these and related results can be found in [OU16]. For simplicity of presentation, we focus on the one-dimensional case.

2 Spectra of Finite Measure

Bounded Spectra. The classical case is when $S = [a, b]$ an interval. Then the elements of PW_S are entire functions of exponential type. The distribution of zeros of such functions is very well studied, see [Le96]. In particular, if the density $D(\Lambda)$ exists, then the condition $D(\Lambda) \geq |S|$ is necessary while the condition $D(\Lambda) > |S|$ is sufficient for Λ to be a uniqueness set for PW_S , where $|S| = b - a$ denotes the measure of S . This can be shown by standard complex variable techniques. A classical result of Beurling and Malliavin [BM67] states that the same is true for irregular sets Λ , provided the uniform density is replaced with a certain exterior one (the Beurling–Malliavin density).

In the case of disconnected spectra S , the uniqueness property of u.d. sets cannot be expressed in terms of their density: Some "dense" (relatively to the measure of S) u.d. sets Λ may be non-uniqueness sets for PW_S . For example, one can easily check that $\Lambda = \mathbb{Z}$ is a non-uniqueness set for PW_S , where $S = [0, \epsilon] \cup [1, 1 + \epsilon]$, $0 < \epsilon < 1$.

On the other hand, some "sparse" u.d. sets Λ may be uniqueness sets for PW_S with a "large" spectrum S . This phenomenon was discovered by Landau [La64], who proved that certain perturbations of \mathbb{Z} produce uniqueness sets for PW_S whenever S is a finite union of intervals $[k + a, k + 1 - a]$, where $0 < a < 1/2$ is any fixed number. The uniqueness sets Λ constructed by Landau have a complicated structure.

A more general result is proved in [OU08]:

Theorem 1. *The set*

$$\Lambda := \{n + 2^{-|n|}, n \in \mathbb{Z}\}$$

is a uniqueness set for PW_S , for every bounded set S satisfying $|S| < 1$.

This theorem remains true for the bounded sets S of arbitrarily large measure satisfying $|S_1| < 1$, where we denote by

$$S_a := (S + a\mathbb{Z}) \cap [0, a] \tag{2}$$

the "projection" of S onto $[0, a]$.

Moreover, the result holds true also for the unbounded sets of finite measure which have a "moderate accumulation" at infinity, see [OU08].

Using re-scaling, one may formulate a corresponding result for any bounded set S .

Unbounded Spectra of Finite Measure. It was shown in [OU11] (see also [OU16], Lec. 10) that for every (bounded or unbounded) set S in \mathbb{R} of finite measure, the space PW_S possesses a u. d. uniqueness set:

Theorem 2. *For every set S of finite measure, there is a u.d. set Λ satisfying $D(\Lambda) = |S|$, which is a uniqueness set for PW_S .*

By the discussion above, the density condition $D(\Lambda) = |S|$ is optimal, since one cannot get a smaller density when S is an interval.

3 Sobolev Spaces with Periodic Spectral Gaps

3.1 Periodic Spectral Gaps

We say that S has periodic "strong" gaps if there exists $a > 0$ such that

$$|\overline{S_a}| < a, \quad (3)$$

where $\overline{S_a}$ denotes the closure of S_a , and the set S_a is defined in (2). Condition (3) means that there is a non-empty interval $I \subset [0, a]$ such that $S \cap (I + a\mathbb{Z}) = \emptyset$.

We say that S has periodic "weak" gaps if

$$|S_a| < a. \quad (4)$$

Condition (4) means that there is a set of positive measure $Q \subset [0, a]$ such that $S \cap (Q + a\mathbb{Z}) = \emptyset$.

Observe that *every set S of finite measure has periodic weak gaps*, since we have $|S_a| < a$, for every $a > |S|$.

3.2 Uniqueness Sets for Sobolev Spaces

Given any u.d. set Λ , it is obvious that there is a non-trivial smooth function f which vanishes on Λ . However, this is no longer so if the spectrum of f has weak periodic gaps. We will state the result for Sobolev spaces.

For every number $\alpha > 1/2$, we denote by $W^{(\alpha)}$ the Sobolev space of functions f such that the Fourier transform $F = \hat{f}$ satisfies

$$\|F\|_\alpha^2 := \int_{\mathbb{R}} (1 + |t|^{2\alpha}) |F(t)|^2 dt < \infty. \quad (5)$$

It is clear that the functions F satisfying (5) belong to $L^1(\mathbb{R})$, and so $W^{(\alpha)}$ consists of continuous functions. We denote by $W_S^{(\alpha)}$ the subspace of $W^{(\alpha)}$ of functions f with spectrum in S , i.e. $F = 0$ a.e. outside S .

Theorem 3. *Suppose a set S satisfies $|S_a| < a$, for some $a > 0$. Then there is a u.d. set Λ of density $D(\Lambda) = a$, which is a uniqueness set for $W_S^{(\alpha)}$, $\alpha > 1/2$.*

3.3 Decomposition of \mathbb{Z}

Lemma 1. *Let $A \subset [0, 1]$, $|A| < 1$. There exist pairwise disjoint sets $Z_j \subset \mathbb{Z}$, $j \in \mathbb{N}$, such that every exponential system*

$$\{e^{-i2\pi nt}, n \in Z_j\} \quad (6)$$

is complete in $L^2(A)$.

Proof. 1. Observe that the exponential family

$$\{e^{-i2\pi nt}, |n| > N\} \quad (7)$$

is complete in $L^2(A)$, for every natural N . Indeed, assume there exists a non-trivial function $F \in L^2(A)$ orthogonal to the system (7). Extend F by zero to $[0, 1] \setminus A$. Then F is orthogonal to the system (7) in $L^2(0, 1)$. Since the trigonometrical system forms an orthonormal basis in $L^2(0, 1)$, we conclude that F is a trigonometric polynomial:

$$F(t) = \sum_{|n| \leq N} c_j e^{-i2\pi nt}.$$

Clearly, F cannot vanish on the set of positive measure $[0, 1] \setminus A$, which is a contradiction.

2. Fix a sequence $\epsilon_k, k \in \mathbb{N}$, satisfying $\epsilon_k \rightarrow 0, k \rightarrow \infty$. We will now construct a sequence of disjoint finite symmetric sets $\Gamma_k \subset \mathbb{Z}, k \in \mathbb{N}$, with the following property: For every $|m| \leq k$, there is a trigonometric polynomial $P_{k,m}$ whose frequencies belong to Γ_k , such that

$$\|e^{i2\pi mt} - P_{k,m}(t)\|_{L^2(A)} < \epsilon_k. \quad (8)$$

Set $\Gamma_1 := \{-1, 0, 1\}$. Clearly, (8) holds with $m = 0, -1, 1$. Then set

$$\Gamma_k := \{n : n_{k-1} < |n| \leq n_k\},$$

where $n_1 = 1$, and we choose $n_j, j > 1$, inductively as follows: By Step 1, there exists n_2 so large that for every $|m| \leq 2$ there is a polynomial $P_{2,m}$ satisfying (8) with $k = 2$ and whose frequencies belong to the set Γ_2 , and so on. On the k -th step, we choose an integer n_k so large that for every $|m| \leq k$ there is a polynomial $P_{k,m}$ satisfying (8) and whose frequencies belong to the set Γ_k .

3. Now, take a partition of \mathbb{N} into disjoint infinite subsets Δ_j and set

$$Z_j := \bigcup_{k \in \Delta_j} \Gamma_k.$$

It follows from the construction above that every exponential system (6) is complete in $L^2(A)$. \square

Remark 1. *It is easy to see that one may construct the sets Z_j so that*

$$D(\cup_{j=1}^{\infty} Z_j) = 1.$$

3.4 Periodization and Fourier Transform

For an integrable function H on the circle group $\mathbb{T} := \mathbb{R}/\mathbb{Z}$, we denote by

$$c_n(H) := \int_{\mathbb{T}} H(t) e^{2\pi i n t} dt, \quad n \in \mathbb{Z},$$

the Fourier coefficients of H .

Given $F \in L^1(\mathbb{R})$, consider its "periodization"

$$H(u) := \sum_{k \in \mathbb{Z}} F(u + k), \quad u \in [0, 1].$$

Clearly, H is defined a.e. and belongs to $L^1(\mathbb{T})$. Direct calculation shows that its Fourier coefficients satisfy $c_n(H) = f(n)$, where f is the inverse Fourier transform of F .

Similarly, for the periodization H_v of the function

$$F_v(t) := e^{2\pi i v t} F(t),$$

we have

$$c_n(H_v) = f(n + v), \quad n \in \mathbb{Z}. \quad (9)$$

It is easy to check that the periodization of an L^2 -function does not always belong to $L^2(\mathbb{T})$. However, the following is true:

Lemma 2. *Assume F satisfies $\|F\|_\alpha < \infty$. Then*

$$\int_0^1 |H(t)|^2 dt < \infty.$$

Proof. Indeed, we have

$$|H(t)|^2 = \left| \sum_{n \in \mathbb{Z}} F(t + na) \right|^2 \leq \sum_{n \in \mathbb{Z}} \frac{1}{(1 + |n|^\alpha)^2} \sum_{n \in \mathbb{Z}} |F(t + na)|^2 (1 + |n|^\alpha)^2,$$

and the lemma easily follows from the definition of $\|F\|_\alpha$ in (5). \square

3.5 Proof of Theorem 3

By re-scaling we can assume that $a = 1$.

Using Lemma 1 with $A = S_1$, write $\mathbb{Z} = \bigcup_{j=1}^\infty Z_j$, where each exponential system (6) is complete in $L^2(S_1)$. It means that each Z_j is a uniqueness set for the space PW_{S_1} .

Fix a sequence $\{\alpha_j\}$ dense in $[0, 1]$, and set

$$\Lambda := \bigcup_{j=1}^\infty (Z_j + \alpha_j). \quad (10)$$

We may assume that Λ is u.d. and $D(\Lambda) = 1$.

Now we will prove that Λ is a uniqueness set for the space $W_S^{(\alpha)}$. We have to show that every function $f \in W_S^{(\alpha)}$ satisfying

$$f|_\Lambda = 0 \tag{11}$$

must vanish on \mathbb{R} .

Let $F := \hat{f}$. Fix $j \in \mathbb{N}$ and consider the function

$$F_j(t) := e^{2\pi\alpha_j t} F(t),$$

and its periodization H_j . Recall that F vanishes a.e. outside S . Since $S \subset S_1 + \mathbb{Z}$, we have

$$H_j = 0 \quad \text{a.e. on } \mathbb{T} \setminus S_1.$$

Also, by Lemma 2, $H_j \in L^2(\mathbb{T})$.

By (9), (10) and (11),

$$c_n(H_j) = f(\alpha_j + n) = 0, \quad n \in Z_j.$$

Since Z_j is a uniqueness set for PW_{S_1} , we have $H_j = 0$ a.e. By (9), this means that

$$f(n + \alpha_j) = 0, \quad n \in \mathbb{Z}.$$

Since this equality is true for all j , f is continuous and the sequence $\{\alpha_j\}$ is dense in $[0, 1]$, we conclude that $f = 0$.

4 Uniqueness Sets for Fast Decreasing Functions

Theorem 3 shows that certain classes of smooth functions f having periodic weak spectral gaps admit u.d. uniqueness sets. In this section we show that a similar result holds for functions f whose Fourier transforms F are smooth functions.

Let us denote by Y the space of continuous functions f satisfying

$$\sup_{x \in \mathbb{R}} (1 + x^2) |f(x)| < \infty. \tag{12}$$

Denote by Y_S the subspace of Y of functions f such that $F = \hat{f} = 0$ outside S .

Theorem 4. *Suppose a set S satisfies $|S_a| < a$, for some $a > 0$. Then there is a u.d. set Λ of density $D(\Lambda) = a$, which is a uniqueness set for Y_S .*

The proof below shows that in Theorem 4 condition (12) in the definition of Y can be somewhat relaxed. However, the result is no longer true if no decay condition is imposed, see Theorem 6 below.

4.1 Proof of Theorem 4

The proof follows the same idea used in the proof of Theorem 3. However, the periodization of F cannot be defined pointwisely.

We will use the following corollary of the classical Poisson summation formula:

Lemma 3. Assume a continuous function f satisfies (12) and $\hat{f}(t) = 0, t \in Q + \mathbb{Z}$, for some set $Q \subset [0, 1], |Q| > 0$. Then for every $x \in [0, 1]$ we have

$$\sum_{n \in \mathbb{Z}} f(x + n) e^{-i2\pi nt} = 0, \quad t \in Q. \quad (13)$$

Proof. If $F = \hat{f}$ is also fast decreasing, then this claim follows directly from the Poisson formula.

Otherwise, apply the Poisson formula to the convolution $(f * h_\epsilon)(x)$, where

$$h_\epsilon := \left(\frac{1}{2\epsilon} \mathbf{1}_{(-\epsilon, \epsilon)} \right)^{2*}$$

and $\mathbf{1}_{(-\epsilon, \epsilon)}$ is the indicator function of $(-\epsilon, \epsilon)$:

$$\sum_{n \in \mathbb{Z}} (f * h_\epsilon)(x + n) e^{-i2\pi nx} = 0, \quad x \in [0, 1], \quad t \in Q.$$

Now, we claim:

$$\sum_{n \in \mathbb{Z}} (f * h_\epsilon)(x + n) e^{-i2\pi nt} \rightarrow \sum_{n \in \mathbb{Z}} f(x + n) e^{-i2\pi nt}$$

as $\epsilon \rightarrow 0$, which proves the lemma. Indeed, fix $\delta > 0$ and decompose the left side into two sums: $\sum_{|n| < N} + \sum_{|n| \geq N}$. One can choose $N = N(\delta)$ so that modulus of the second summand is $< \delta$, for every $x, t \in [0, 1]$ and $0 < \epsilon < 1$. Clearly, each term of the first summand goes to $f(x) e^{-i2\pi nt}$ as $\epsilon \rightarrow 0$, due to the continuity of f . \square

Now, we can finish the proof of Theorem 4. By re-scaling, we may assume that $a = 1$, so that $|S_1| < 1$.

Following the proof of Theorem 3, we may find pairwise disjoint sets $Z_j \subset \mathbb{Z}, j \in \mathbb{N}$, such that for every j the system

$$E(Z_l) := \{e^{-2\pi ikt}, k \in Z_l\}$$

is complete in $L^2(S_1)$.

Set

$$\Lambda := \cup_{j \in \mathbb{N}} (Z_j + \alpha_j),$$

where $\{\alpha_l, l \in \mathbb{N}\}$ is dense in $(0, 1)$. It remains to check that Λ is a uniqueness set for Y_S .

Assume $f|_\Lambda = 0$, for some $f \in Y_S$, i.e. we have

$$f|_{Z_j + \alpha_j} = 0, \quad j = 1, 2, \dots$$

Fix j and consider a 1-periodic function

$$g_j(x) := \sum_{n \in \mathbb{Z}} f(n + \alpha_j) e^{-i2\pi nx}.$$

Clearly, $g \in L^2(0, 1)$ and is orthogonal in $L^2(0, 1)$ to all the exponential functions in $E(Z_j)$. On the other hand, due to Lemma 3,

$$g_j(x) = 0, \quad t \in Q := [0, 1] \setminus S_1.$$

The completeness of $E(Z_j)$ in $L^2(S_1)$ implies that $g_j = 0$ a.e. Hence,

$$f(n + \alpha_j) = 0, \quad n \in \mathbb{Z}.$$

This is true for every j . Recalling that $\{\alpha_j\}$ is dense on $[0, 1]$ and $f \in C(\mathbb{R})$, we conclude that $f = 0$ on \mathbb{R} .

5 Distributions with Periodic Spectral Gaps

5.1 Strong Gaps

If S has periodic strong gaps, then the results above can be extended to wider function spaces.

Denote by X the space of continuous functions that have at most polynomial growth on \mathbb{R} . Every element $f \in X$ is a Schwartz distribution. Its spectrum is the minimal closed set S such that for every test function φ satisfying $\hat{\varphi} = 0$ in a neighbourhood of S , we have

$$\int_{\mathbb{R}} f(t) \varphi(t) dt = 0.$$

Given closed set S , we denote by X_S the subspace of X consisting of functions with spectrum in S .

Without loss of generality, we may assume the spectral gaps are $[0, \delta] + \mathbb{Z}$.

Theorem 5. *There is a u.d. set Λ , $D(\Lambda) = 1$, which is a uniqueness set for X_S , $S = [0, 1 - \delta] + \mathbb{Z}$, for every $0 < \delta < 1$.*

Proof. Consider Z_j as in Lemma 1 (this can be done independently on δ). Choose Λ as in the proof of Theorem 4. Given $f \in X_S$, consider the function $g := f \cdot \varphi$, where φ is a Schwartz function supported by $[0, \delta/2]$.

It is easy to see that g satisfies the assumptions of Theorem 4 with $a = 1$, $S = [0, 1 - \delta/2]$. If $f|_{\Lambda} = 0$, then the same is true for g . So, Theorem 4 implies $f = 0$. \square

5.2 Weak Gaps

Here we show that Theorem 5 is no longer true for the weak spectral gaps. This is a direct corollary of a result from [OU09].

We need the following

Definition 1. *Given a closed (not necessarily bounded) set S , the Bernstein space B_S is the set of continuous bounded functions f on \mathbb{R} whose spectrum (in distributional sense) lies in S .*

Theorem 6. ([OU09]) *There is a closed set S of Lebesgue measure zero such that every bounded function $c(\lambda)$ defined on a u.d. set Λ can be interpolated by a function $f \in B_S$.*

It is obvious that every set of measure zero has weak periodic gaps with an arbitrary period a . However, no u.d. set Λ is a uniqueness set for B_S .

A few words about the proof of Theorem 6. It is based on a classical result of D.E. Menshov (1916) (see [Ba64]): *There is a probability measure μ on \mathbb{R} supported by a compact set K of measure zero, and such that its Fourier transform*

$$\hat{\mu}(x) = \int_K e^{-2\pi it} d\mu(t)$$

vanishes at infinity.

Here is a short sketch of the proof (see details in [OU16], Lec. 10).

Proof. 1. Given δ , by re-scaling one can get a probability measure μ_δ supported by a compact K of Lebesgue measure zero, such that

$$\hat{\mu}_\delta(0) = 1, |\hat{\mu}_\delta(x)| < \delta, \quad |x| > \delta.$$

2. Using this, one can construct a family of compact sets K_j of measure zero, which goes to infinity, and functions $g_j \in B_{K_j}$, $j \in \mathbb{N}$, satisfying

$$\|g_j\|_\infty = g_j(0) = 1, |g_j(t)| < e^{-j}, \quad |t| > e^{-j}.$$

3. Set

$$S := \cup_{j=1}^\infty K_j.$$

It is a closed (non-compact) set of measure zero.

Fix any $\delta > 0$ and any u.d. set Λ . Using appropriate translates of the functions g_j , one can define functions $f_\lambda \in B_S$ satisfying

$$\|f_\lambda\|_\infty = f_j(\lambda) = 1, |f(\lambda')| < e^{-2|\lambda' - \lambda|/\delta}, \quad \lambda_j \in \Lambda, \lambda_l \neq \lambda_j.$$

4. Consider the linear operator $T : l^\infty(\Lambda) \rightarrow l^\infty(\Lambda)$ defined by

$$(Tc)_\lambda := \sum_{\lambda' \in \Lambda, \lambda' \neq \lambda} f_{\lambda'}(\lambda) c_{\lambda'}, \quad \lambda \in \Lambda, \quad c = \{c_{\lambda'}, \lambda' \in \Lambda\} \in l^\infty(\Lambda).$$

Clearly, $\|T\| < 1$. Hence, the operator $T+I$ is surjective. Therefore, for every datum $c = \{c_\lambda\} \in l^\infty(\Lambda)$ there is a sequence $b = \{b_\lambda\} \in l^\infty(\Lambda)$ satisfying $(I+T)b = c$. Hence, the function

$$f(x) := \sum_{\lambda \in \Lambda} b_\lambda f_\lambda(x).$$

belongs to B_S and solves the interpolation problem $f|_\Lambda = c$. □

6 Non-Periodic Spectral Gaps

Here we show that the periodicity of spectral gaps is crucial for existence of discrete uniqueness sets.

Let us consider spectra S which are unions of disjoint intervals of a given length. For simplicity, we assume that each interval has length one:

$$S = \cup_{j=1}^\infty [\gamma_j, \gamma_j + 1]. \tag{14}$$

We also assume that the distances ξ_j between the intervals belong to a fixed interval, say $[2, 3]$:

$$\xi_j := \gamma_{j+1} - \gamma_j - 1 \in [2, 3], \quad j \in \mathbb{N}. \quad (15)$$

Clearly, S belongs to a half-line $[\gamma_1, \infty)$ and admits a representation

$$S = \Gamma + [0, 1], \quad \Gamma := \cup_{j=1}^{\infty} \{\gamma_j\}, \quad (16)$$

where Γ satisfies $\delta(\Gamma) \geq 3$. Here $\delta(\Gamma)$ is the separation constant defined in (1).

Now we introduce a certain property of u.d. sets. We say that u.d. set Γ satisfies property (C) if it contains arbitrary long arithmetic progressions with rationally independent steps. More precisely, we assume

(C) For every $m \in \mathbb{N}$ there are rationally independent numbers q_1, \dots, q_m , such that for every $N \in \mathbb{N}$ the set Γ contains arithmetic progressions of length N with differences q_1, \dots, q_m . The latter means that there exist a_1, \dots, a_m such that

$$\bigcup_{j=1}^m \{a_j + q_j, a_j + 2q_j, \dots, a_j + Nq_j\} \subset \Gamma.$$

Theorem 7. *Assume S is given in (14)–(16), where Γ satisfies property (C). Then no u.d. set Λ is a uniqueness set for the Sobolev space W_S^α .*

Below we prove this result in a stronger form.

One may also check that, under the assumptions of Theorem 7, no u.d. set Λ is a uniqueness set for the space Y_S .

6.1 Interpolation Sets

A set Λ is called an *interpolation set* for the Paley-Wiener space PW_S , if for every sequence $\{c_\lambda, \lambda \in \Lambda\} \in l^2(\Lambda)$ there exists $f \in PW_S$ satisfying

$$f(\lambda) = c_\lambda, \quad \lambda \in \Lambda.$$

The following criteria is well-known (see e.g. [OU16], Lec. 4):

Lemma 4. *Let S be a bounded set and Λ a u.d. set. Then Λ is a set of interpolation for PW_S if and only if there is a constant $C > 0$ such that the inequality*

$$\int_S \left| \sum_{\lambda \in \Lambda} c_\lambda e^{i2\pi\lambda t} \right|^2 dt \geq C \sum_{\lambda \in \Lambda} |c_\lambda|^2$$

holds for every finite sequence c_λ .

Theorem 7 is a direct corollary of the following

Main Lemma. *Assume S is a set from Theorem 7. Then for every $\delta > 0$ there is a bounded subset $S(\delta) \subset S$ such that every u.d. set Λ satisfying $\delta(\Lambda) \geq \delta$ is a set of interpolation for $PW_{S(\delta)}$.*

Indeed, consider a set $\Lambda \cup \{c\}$ for some point $c \notin \Lambda$. By the Main Lemma, it is a set of interpolation for $PW_{S(\delta)}$, for some bounded subset $S(\delta) \subset S$. Then there exists $f \in PW_{S(\delta)}$ satisfying $f(c) = 1$ and

$$f(\lambda) = 0, \quad \lambda \in \Lambda.$$

It remains to observe that $PW_{S(\delta)} \subset W_S^\alpha$.

6.2 Proof of Main Lemma

Lemma 5. *Suppose Γ satisfies property (C). Then for every $0 < \epsilon < 1$ there exist $N \in \mathbb{N}$ and $\eta_j \in \Gamma, j = 1, \dots, N$, such that the exponential polynomial*

$$P(t) = \frac{1}{N} \sum_{j=1}^N e^{i\eta_j t}$$

satisfies

$$|P(t)| \leq \epsilon, \quad \epsilon < |t| < 1/\epsilon. \quad (17)$$

Proof. 1. Fix any integer $m > 1/\epsilon$. Then fix numbers q_1, \dots, q_m in the definition of property (C). Since q_j are rationally independent, the set of points

$$\{kq_j \in (-1/\epsilon, 1/\epsilon)\} \quad (18)$$

is separated, where $k \in \mathbb{Z}, k \neq 0, j = 1, \dots, m$. Hence, the distance between any two points in this set exceeds some positive number ρ . We may assume that $\rho < \epsilon$.

2. For $n \in \mathbb{N}$ and $q \geq 2$, consider the $(1/q)$ -periodic exponential polynomial

$$P_{n,q}(t) := \frac{1}{n} \sum_{j=0}^{n-1} e^{i2\pi jqt} = \frac{1}{n} \frac{e^{i2\pi nqt} - 1}{e^{i2\pi qt} - 1}.$$

From the properties of Dirichlet kernel, it is well-known that it satisfies

$$|P_{n,q}(t)| \leq \rho, \quad \text{dist}(t, (1/q)\mathbb{Z}) \geq \rho, \quad (19)$$

provided n is large enough.

3. Choose n so large that (19) holds with $q = q_j, j = 1, \dots, m$. Then, since the set (18) is ρ -separated, for every t satisfying $\epsilon < |t| < 1/\epsilon$, the inequality

$$|P_{n,q_j}(t)| < \epsilon$$

holds for all but at most one value of $j \in \{1, \dots, m\}$.

4. By the definition of property (C), there exist a_j such that $a_j + kq_j \in \Gamma, k = 0, \dots, n-1$. Set

$$P(t) = \frac{1}{m} \sum_{j=1}^m e^{i2\pi a_j t} P_{n,q_j}(t).$$

By Step 3, we see that

$$|P(t)| < \frac{1 + (m-1)\epsilon}{m} < \epsilon, \quad \epsilon < |t| < 1/\epsilon,$$

which completes the proof. \square

Proof of Main Lemma. Fix $\delta > 0$ and assume that a u.d. set Λ satisfies $\delta(\Lambda) \geq \delta$.

By Lemma 5, for every $0 < \epsilon < 1$ there is an exponential polynomial P with frequencies in Γ satisfying (17). We denote the set of its frequencies by $\Gamma_P \subset \Gamma$, and set

$$S(\delta) := \Gamma_P + [0, 1].$$

Clearly, $S(\delta)$ is a bounded subset of S .

Now we fix any positive smooth function Φ which vanishes outside $[0, 1]$ such that its Fourier transform $\varphi = \hat{\Phi}$ satisfies $\varphi(0) = 1$ and

$$\sup_{x \in \mathbb{R}} (1 + x^4) |\varphi(x)| < \infty. \quad (20)$$

Set

$$H(t) := (\Phi * \sum_{\gamma \in \Gamma_P} \delta_\gamma)(t) = \sum_{\gamma \in \Gamma_P} \Phi(t - \gamma).$$

Then the support of H belongs to $S(\delta)$ and its Fourier transform is given by

$$h(x) := \hat{H}(x) = P(x)\varphi(x).$$

Clearly, $h(0) = 1$.

When ϵ is sufficiently small, from (17) and (20) we get

$$|h(x)| < \frac{\epsilon}{1 + x^2}, \text{ for all } |x| > \delta,$$

where δ is the separation constant of Λ . Using this estimate and assuming that ϵ is sufficiently small, for every $\lambda \in \Lambda$ we get the estimate

$$\sum_{\mu \in \Lambda, \mu \neq \lambda} |h(\mu - \lambda)| < \sum_{\mu \in \Lambda, \mu \neq \lambda} \frac{\epsilon}{1 + (\mu - \lambda)^2} < 2 \sum_{n \in \mathbb{N}} \frac{\epsilon}{1 + (\delta n)^2} < \frac{1}{2}.$$

Set

$$M := \max_{t \in [0, 1]} |\Phi(t)|.$$

Then

$$\begin{aligned} \int_{S(\delta)} \left| \sum_{\lambda \in \Lambda} c_\lambda e^{i\lambda t} \right|^2 dt &\geq \frac{1}{M} \int_{S(\delta)} \left| \sum_{\lambda \in \Lambda} c_\lambda e^{i\lambda t} \right|^2 H(t) dt \\ &= \frac{1}{M} \left(\sum_{\lambda \in \Lambda} |c_\lambda|^2 + \sum_{\lambda, \mu \in \Lambda, \lambda \neq \mu} c_\lambda \bar{c}_\mu h(\lambda - \mu) \right) \geq \\ &\frac{1}{M} \left(\sum_{\lambda \in \Lambda} |c_\lambda|^2 - \sum_{\lambda, \mu \in \Lambda, \lambda \neq \mu} \frac{|c_\lambda|^2 + |c_\mu|^2}{2} |h(\lambda - \mu)| \right) \geq \\ &= \frac{1}{M} \left(\sum_{\lambda \in \Lambda} |c_\lambda|^2 - \sum_{\lambda \in \Lambda} |c_\lambda|^2 \sum_{\lambda, \mu \in \Lambda, \mu \neq \lambda} |h(\lambda - \mu)| \right) > \frac{1}{2M} \sum_{\lambda \in \Lambda} |c_\lambda|^2. \end{aligned}$$

By Lemma 4, this completes the proof. \square

6.3 Random Spectra do not Admit u.d. Uniqueness Sets

Here we consider the situation when S is a countable union of unite intervals, the distances between the intervals being randomly distributed. More precisely, below we assume that S and Γ are defined in (14) and (16), and that the $\xi_j := \gamma_{j+1} - \gamma_j - 1$ are independent random variables uniformly distributed over the interval $[2, 3]$. With these assumptions, we have

Theorem 8. *With probably one no u.d. set Λ is a uniqueness set for $W_S^{(\alpha)}$.*

Proof. Theorem 8 follows from the following claim, which is an analogue of the Main Lemma in Sec. 6.1: *With probability one, for every fixed $\delta > 0$ there is a bounded subset $S(\delta) \subset S$ such that every u.d. set $\Lambda, \delta(\Lambda) \geq \delta$, is a set of interpolation for $PW_{S(\delta)}$.*

Recall that

$$S = \cup_{j=1}^{\infty} \{\gamma_j\} + [0, 1], \quad \gamma_{j+1} - \gamma_j \in [3, 4], \quad j \in \mathbb{N}.$$

It is easy to see that given any integers $k \geq 1, N \geq 2$ and number $q \in (3, 4)$, the set

$$\{\gamma_k, \gamma_k + q, \dots, \gamma_k + Nq\} + [1/4, 3/4]$$

belongs to S whenever

$$|\gamma_{k+j} - (\gamma_k + jq)| < \frac{1}{4}, \quad j = 1, 2, \dots, N.$$

Recall also that $\gamma_{j+1} - \gamma_j$ is uniformly distributed over $[3, 4]$. So, the probability that the latter inequalities hold true is positive and independent on k .

Now, fix any $m \in \mathbb{N}$ and $q_1, \dots, q_m \in (3, 4)$. By the the Borel-Cantelli lemma, one can see that with probability one there are integers k_1, \dots, k_m such that the finite sequence

$$\Gamma^* := \bigcup_{j=1}^m \{\gamma_{k_j}, \gamma_{k_j} + q_j, \dots, \gamma_{k_j} + Nq_j\}$$

satisfies

$$S(\delta) := \Gamma^* + [1/4, 3/4] \subset S.$$

Now, choosing m and N sufficiently large, the claim above follows exactly the same way as in the proof of the Main Lemma. \square

7 Remarks

7.1 Multi-dimensional Extensions

All our one-dimensional results above admit multi-dimensional extensions. Here we give a very brief account of these extensions.

The definitions in Sec. 1 can be extended to the multi-dimensional situation. In particular, given a set $S \subset \mathbb{R}^p$, the Paley-Wiener space PW_S consists of the (p -dimensional) inverse Fourier transforms of the $L^2(\mathbb{R}^p)$ -functions which vanish a.e.

outside S . A set $\Lambda \subset \mathbb{R}^p$ is uniformly discrete (u.d.), if the infimal distance between its different elements is positive. A u.d. set Λ possesses a uniform density $D(\Lambda)$ if

$$\text{Card}(\Lambda \cap ([0, r]^p + s)) = r^p D(\Lambda) + o(r^p) \text{ uniformly on } s \text{ as } r \rightarrow \infty.$$

Here $s = (s_1, \dots, s_p) \in \mathbb{R}^p$ and

$$[0, r]^p + s = \{x = (x_1, \dots, x_p) \in \mathbb{R}^p : s_j \leq x_j \leq s_j + r, j = 1, \dots, p\}.$$

We will denote by $|S|$ the p -dimensional measure of a set $S \subset \mathbb{R}^p$.

Denote by S_a , where a is a positive number, the "projection" of a set $S \subset \mathbb{R}^p$ onto the cube $[0, a]^p$:

$$S_a := (S + a\mathbb{Z}^p) \cap [0, a]^p.$$

We will now formulate a multi-dimensional analogue of Theorem 1: *Suppose that s_1, \dots, s_p are real numbers linearly independent over the set of integers. Then the set*

$$\Lambda := \{m_1 + s_1 2^{-|m_1| - \dots - |m_p|}, \dots, m_p + s_p 2^{-|m_1| - \dots - |m_p|}, (m_1, \dots, m_p) \in \mathbb{Z}^p\}$$

is a uniqueness set for PW_S , for every bounded set $S \subset \mathbb{R}^n$ satisfying $|S_1| < 1$.

The proof of this result goes on the same lines as the proof of Theorem 2 in [U101].

Choosing the numbers s_j small, one can make the set Λ in the above result an arbitrarily small perturbation of the lattice \mathbb{Z}^p .

Also Theorem 2, as noted in [OU11], admits an extension to several dimensions: *For every set $S \subset \mathbb{R}^p$ of finite measure there is a u.d. set $\Lambda \subset \mathbb{R}^p$, $D(\Lambda) = |S|$, which is a uniqueness set for PW_S .*

In order to get multi-dimensional versions of Theorems 3 and 4, one may introduce multi-dimensional analogues of the spaces $W_S^{(\alpha)}$ and Y_S as follows: The space $W^{(\alpha)}(\mathbb{R}^p)$, $\alpha > p/2$, consists of functions f defined on \mathbb{R}^p which are the Fourier transform of functions F vanishing outside S and satisfying

$$\|F\|^2 := \int_{\mathbb{R}^p} (1 + |t|^{2\alpha}) |F(t)|^2 dt < \infty,$$

where $|t|^2 = t_1^2 + \dots + t_p^2$ and $dt = dt_1 \cdot \dots \cdot dt_p$.

The space $Y_S(\mathbb{R}^p)$ consists of continuous functions f satisfying

$$\sup_{x \in \mathbb{R}^p} (1 + |x|^{2p}) |f(x)| < \infty, \quad |x|^2 := x_1^2 + \dots + x_p^2,$$

and such that the Fourier transform \hat{f} vanishes outside S . One may check that both Lemmas 1 and 2 above admit multi-dimensional extensions. This allows to get multi-dimensional analogues of Theorems 3 and 4: *Assume $S \subset \mathbb{R}^p$ is such that $|S_a| < a^p$, for some $a > 0$. Then the spaces $W_S^{(\alpha)}(\mathbb{R}^p)$, $\alpha > p/2$, and $Y_S(\mathbb{R}^p)$ admit a u.d. uniqueness set.*

One may also check that the p -dimensional versions of Theorems 5–8 hold true.

7.2 Questions

We leave open several problems which might be of certain interest.

1. In connection with Theorems 1 and 2, one may ask: *Does there exist a u.d. set Λ , $D(\Lambda) = 1$, which is a uniqueness set for PW_S , for every set $S \subset \mathbb{R}$, $|S| < 1$?*
2. The following question arises in connection with Theorems 3 and 4: *Let $S \subset \mathbb{R}$ be a set with periodic weak gaps. Does the space $PW_S \cap C(\mathbb{R})$ admit a u.d. uniqueness set?*
3. It also seems an interesting question if Theorem 2 remains true for the Fourier transforms of integrable functions: *Let $S \subset \mathbb{R}$ be a set of finite measure. Is it true that the space*

$$\widehat{L}_S := \{f = \hat{F} : F \in L^1(\mathbb{R}), F = 0 \text{ a.e. outside } S\}$$

admits a u.d. uniqueness set?

Theorem 5 implies that the answer is "yes" whenever S_a is not dense on $[0, a]$, for some a .

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